Second derivative of cost function and $H^1$ Newton method for topology optimization problem of density type

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Abstract

Problems making the optimum holes in domains of boundary value problems with respect to partial differential equations have been called topology optimization problems. Among them, a problem in which a density is defined as a design variable has been classified into a topology optimization problem of density type. So far, to find descent directions of cost functions, the $H^1$ gradient method has been used. However, low convergence has been made a problem. In this study, we derived the second-order Fréchet derivatives of cost functions for the topology optimization problem, and developed a method seeking descent directions of cost functions by a Newton method ($H^1$ Newton method). Effectiveness of the method was demonstrated with numerical results.

The topology optimization problem of density type is formulated in the following way. Let $D$ be a $d \in \{2,3\}$ dimensional Lipschitz domain. We define a design variable by $\theta \in X = H^1(D; \mathbb{R})$, and a density by its sigmoid function as $\phi(\theta) = (\tanh \theta + 1)/2$. A boundary value problem of partial differential equation is formulated by using an exponential function of $\phi(\theta)$ as a coefficient of the partial differential equation. An objective function $f_0$ and constraint functions $f_1, \cdots, f_m$ are defined by functionals of $\phi(\theta)$ and the solution of the boundary value problem. With respect to $i \in \{0,1,\cdots,m\}$, $\theta$-derivative $f'_i(\theta)[\vartheta] = \langle g_i, \vartheta \rangle$ of $f_i$ is obtained by the Lagrange multiplier method with respect to the boundary value problem of partial differential equation. $\theta$-Hessian $f''_i(\theta)[\vartheta_1, \vartheta_2] = h_i(\theta_k)[\vartheta_1, \vartheta_2]$ of $f_i$ is derived using the second-order Fréchet derivative of the Lagrange function for $f_i$. A proposed Newton method for the topology optimization problem of density type is as follows. With respect to an iteration number $k \in \{0,1,\cdots\}$, we define a Hessian of the Lagrange function $L$ for the topology optimization problem of density type as

$$h_{\mathcal{L}}(\theta_k)[\vartheta_1, \vartheta_2] = h_0(\theta_k)[\vartheta_1, \vartheta_2] + \sum_{i \in \{1,\cdots,m\}} \lambda_i h_i(\theta_k)[\vartheta_1, \vartheta_2],$$

where $\lambda_i$ is the Lagrange multiplier for $f_i \leq 0$. Moreover, let $a_X : X \times X \to \mathbb{R}$ be a bilinear form to ensure the coerciveness and boundedness of $h_{\mathcal{L}}$. In this study, we determine a descent direction of $f_i$ by the following equation for an arbitrary $\psi \in X$.

$$h_{\mathcal{L}}(\theta_k)[\vartheta_{gi}, \psi] + a_X(\vartheta_{gi}, \psi) = -\langle g_i(\theta_k), \psi \rangle$$

A computer program for numerical analysis based on the method was made using the FreeFem++ Language. Figure 1 shows the results for a mean compliance minimization problem of a two dimensional linear elastic body.

![Figure 1: Comparison of $H^1$ Newton method and $H^1$ gradient method](image-url)