Smoothing Gradient Method for Topology Optimization Problems of Continua

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We developed a numerical solution to non-parametric topology optimization problems of continua. The problems have been considered as finding problems of the optimum distributions of cavities in domains of continua. For this problem, it has been known that use of the set of characteristic functions having 1 for material points and 0 for voids causes ill-posedness. To compensate, it has been proposed to formulate using density functions allowing intermediate values and to solve the problem by using the finite-element method and the optimization methods. Especially, the SIMP method using an exponential function of density instead of the characteristic function has been used. In the approaches, there have been observed numerical instability phenomena such as checkerboard patterns or mesh-dependencies. In this study, we present the cause of the numerical instabilities by showing the irregularity of the density gradient and a solution that compensates these instabilities by applying a smoothing gradient method in an appropriate functional space. The validity of the solution is confirmed with a numerical example for a linear elastic problem.

Key Words: Topology optimization, Density method, SIMP method, Density gradient, Regularity, $H^1$ gradient method, SQP (Sequential Quadratic Programming), Primal-dual interior point method

1 Introduction

Problems finding the optimum layouts of voids in domains in which boundary value problems of partial differential equations are defined have been called topology optimization problems of continua. To formulate the problems using a set of the characteristic functions having 1 for material points and 0 for voids was unsuccessful because the admissible set is not compact. Then, problems relaxing the 0,1 condition, such as to select parameters of microstructures based on the homogenization method or to use an exponential function of density instead of the characteristic function (SIMP method) have been solved using the finite-element method and optimization methods.

However, in the use of the relaxed methods, there have been observed numerical instabilities such as checkerboard patterns or mesh-dependencies.

In this study, we present the cause of the numerical instabilities and a solution that compensates these instabilities.

2 Theory and method

Let $\rho \in \mathcal{W}$ be a density defined in a bounded domain $\Omega \in \mathbb{R}^d$ ($d = 2, 3$) where $\mathcal{W}$ is defined with a positive constant number $M$ and a small positive constant number $\rho > 0$ by

$$\mathcal{W} = \left\{ \rho \in H^1(\Omega) \left| \rho \leq \rho_0, \|\rho\|_{H^1} \leq M \right. \right\}. \quad (1)$$

By following the SIMP method, a coefficient of a partial differential equation is multiplied by $\rho^p$. In this study, we assume $p$ satisfies

$$1 \leq p < \frac{2d}{d - 2}. \quad (2)$$

Let $u \in U$ be the solution of a boundary value problem and a objective functional $J^0(\rho, u)$ and constraint functionals $J^l(\rho, u)$ ($l = 1, 2, \cdots, m$) be defined by

$$J^0(\rho, u) = \int_\Omega g^0(\rho, u) \, d\Omega. \quad (3)$$

Let us define a topology an optimization problem based on the SIMP method as

$$\min_{\rho \rho_0 \in \mathcal{W} \times U} \left\{ J^0(\rho, u) \left| J^l(\rho, u) \leq 0 (l = 1, 2, \cdots, m) \right. \right\}. \quad (4)$$

Then, we can show that $u_0 \rightarrow u_0 \in U$ for $\rho_0 \rightarrow \rho_0 \in \mathcal{W}$ and for the equations to calculate the gradients (density gradients) $G^0_p[(\cdot, \cdot)] (l = 0, 1, 2, \cdots, m)$ of $J^0(\rho, u)$ with respect to density variation. The cause of the numerical instability phenomena can be explained by $G^0_p \notin H^1(\Omega)$.

In this study, we propose a method to compute regular density variations by applying a gradient method in an appropriate functional space using $G^0_p$. Since $\mathcal{W}$ belongs to $H^1(\Omega)$, we propose a gradient method in $H^1(\Omega)$ ($H^1$ gradient method). In the $H^1$ gradient method, using $G^0_p$ we calculate $\rho^{(l)} \in H^1(\Omega)$ that decrease $J^0(\rho, u)$ ($l = 0, 1, 2, \cdots, m$) by the weak form:

$$b_{H(\Omega)}(\rho^{(l)}), y = - \left( G^0_p, y \right)_{L^2(\Omega)} \quad \forall y \in H^1(\Omega) \quad (5)$$

where $b_{H(\Omega)}(\cdot, \cdot)$ is a coercive bilinear form in $H^1(\Omega)$.

In this study, we developed a program to compute Eq. (5) by using the finite-element method. We used the SQP for functional constraints and the penalty method and the primal-dual interior point method for density constraints $\rho \leq \rho_0 \leq 1$.

3 Numerical example

Figure 1 shows a numerical example of a linear elastic problem. We assumed that the left plain boundary was fixed and the center of the right plain boundary was loaded with a downward force. We selected the mean compliance for $J^0(\rho, u)$ and increment of the mass for $J^1(\rho)$. The limit value of the mass was set as 35% of the mass when $\rho = 1.0$ in $\Omega$. In Fig. 1, the results by the $L^2$- and $H^1$-gradient methods were computed by using $b_{L^2(\Omega)}(\cdot, \cdot)$ and $b_{H^1(\Omega)}(\cdot, \cdot)$ in Eq. (5) respectively. From the results, it can be observed that the numerical instability phenomena were controlled by the $H^1$ gradient method.

![Fig. 1 Numerical example by SIMP topology optimization problem](attachment:image.png)