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Shape optimization of a rubber bushing

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Abstract

The present paper describes a solution to a non-parametric shape optimization problem of a rubber bushing in order to adjust a function of the reaction force with respect to static displacement to a desired function. The main problem is defined as a static hyperelastic problem considering a large deformation and a non-linear constitutive equation. The squared error norm of the work done by compulsory displacement and the volume are chosen as cost functions. The shape derivatives of the cost functions are derived theoretically. An iterative algorithm based on the H^1 gradient method is used to solve the shape optimization problem.

Keywords hyperelastic problem, shape optimization, H^1 gradient method

Research Activity Group Mathematical Design

1. Introduction

The rubber bushing is used as a vibration isolator in vehicle suspension systems in order to prevent the vibration of an engine or the tire from transferring into the guest room. The rubber bushing has been modeled as a largely deforming hyperelastic continuum and following a non-linear constitutive equation. Many equations have been proposed for the constitutive equation using non-linear elastic potentials [1]. Numerical analyses of the rubber bushing using the finite element method have been reported [2, 3].

Moreover, numerical solutions to parametric shape optimization problems of the rubber bushing have been presented [4, 5]. In these studies, in order to adjust a function of the reaction force with respect to static displacement to a desired function, a squared error norm of the reaction force function has been chosen as a cost function.

In the present paper, we present the solution to the non-parametric shape optimization problem of a rubber bushing. Domain variation from an initial domain is chosen as a design variable. The main problem, which we refer to as a boundary value problem of a partial differential equation in which the domain is defined as a design variable, is formulated as a hyperelastic problem considering large deformation and a non-linear constitutive equation. We choose a squared error norm of the work done by compulsory displacement as an objective function and the volume as a constraint function. The shape derivatives of the cost functions are derived theoretically following the standard procedure using the H^1 gradient method [6], but the geometrical and material non-linearities are considered in the present paper.

2. Admissible set of design variables

First, let us define the admissible set of design variables for the shape optimization problem. Let $\Omega_0 \subset \mathbb{R}^d$ be a $d \in \{2, 3\}$-dimensional domain with a Lipschitz boundary, which is denoted by $\partial\Omega_0$. On $\partial\Omega_0$, $\Gamma_{\mathrm{D}0} \subset \partial\Omega_0$ and $\Gamma_{\mathrm{N}0} = \partial\Omega_0 \setminus \bar{\Gamma}_{\mathrm{D}0}$ ($\bar{\Gamma}_{\mathrm{D}0} = \Gamma_{\mathrm{D}0} \cup \partial\Gamma_{\mathrm{D}0}$) denote the Dirichlet boundary and the homogeneous Neumann boundary, respectively.

We assume that Ω_0 is fixed and that the domain is created by continuous one-to-one mapping $\boldsymbol{i} + \boldsymbol{\phi} : \Omega_0 \to \mathbb{R}^d$ as $\Omega(\boldsymbol{\phi}) = \{ (\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in \Omega_0 \}$, where $\boldsymbol{i}$ is used as the identity mapping. In the same manner, the notation $(\,\cdot\,)(\boldsymbol{\phi})$ is used as $\{ (\boldsymbol{i} + \boldsymbol{\phi})(\boldsymbol{x}) \mid \boldsymbol{x} \in (\,\cdot\,)_0 \}$ in the present paper. In order to define the Fréchet derivatives with respect to domain variation, we use

$$X = \left\{ \boldsymbol{\phi} \in H^1\left(\mathbb{R}^d; \mathbb{R}^d\right) \;\middle|\; \boldsymbol{\phi} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D}0} \right\} \tag{1}$$

as the Banach space for $\boldsymbol{\phi}$. In (1), the domain of $\boldsymbol{\phi}$ is extended to $\mathbb{R}^d$ by Calderón's extension theorem. Moreover, in order to maintain the continuous one-to-one mapping property, we define the admissible set of $\boldsymbol{\phi}$ as

$$\mathcal{D} = \{ \boldsymbol{\phi} \in X \cap Y \mid \|\boldsymbol{\phi}\|_Y < \sigma \}, \tag{2}$$

where Y is defined by $W^{1,\infty}\left(\mathbb{R}^d; \mathbb{R}^d\right)$, and $\sigma > 0$ is chosen such that $\boldsymbol{\phi}$ is a bijection.

3. Main problem

For $\boldsymbol{\phi} \in \mathcal{D}$, let us define the main problem. Let $(0, t_{\mathrm{T}}) \subset \mathbb{R}$ be a time domain with a positive constant t_{T}, and let $\boldsymbol{u}_{\mathrm{D}} : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be a given function denoting a quasi-static compulsion displacement, the magnitude of which increases monotonically with respect to $t \in (0, t_{\mathrm{T}})$ at all $\boldsymbol{x} \in \mathbb{R}^d$.

Let $\boldsymbol{u} : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be a displacement obtained as a solution to a hyperelastic problem shown later in Problem 1 (refer for example [7]). In order to construct this problem, we need to define the constitutive equation of the hyperelastic continuum. Let $\boldsymbol{y} = \boldsymbol{i} + \boldsymbol{u} : (0, t_{\mathrm{T}}) \times \mathbb{R}^d \to \mathbb{R}^d$ be the mapping for the large deformation, and

$$\boldsymbol{F}(\boldsymbol{u}) = \left(\frac{\partial y_i}{\partial x_j} \right)_{ij} = \boldsymbol{I} + \left(\frac{\partial u_i}{\partial x_j} \right)_{ij} \tag{3}$$

be the deformation gradient tensor, where $\boldsymbol{I}$ denotes the unit matrix of d-th order. Using the definition, the Green-Lagrange strain is defined as

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{u}) = (e_{ij}(\boldsymbol{u}))_{ij} &= \frac{1}{2}\left(\boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{I}\right)\\ &= \boldsymbol{E}_{\mathrm{L}}(\boldsymbol{u}) + \frac{1}{2}\boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{u}),\end{aligned} \tag{4}$$

where

$$\boldsymbol{E}_{\mathrm{L}}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)_{ij},$$

$$\boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{2}\left(\sum_{k\in\{1,\cdots,d\}}\frac{\partial u_k}{\partial x_i}\frac{\partial v_k}{\partial x_j}\right)_{ij}.$$

The constitutive equation for hyperelastic material is defined by assuming the existence of a nonlinear elastic potential $\pi : \mathbb{R}^{d\times d} \to \mathbb{R}$ that gives the second Piola-Kirchhoff stress tensor as

$$\boldsymbol{S}(\boldsymbol{u}) = 2\frac{\partial\pi(\boldsymbol{C}(\boldsymbol{u}))}{\partial\boldsymbol{C}(\boldsymbol{u})}. \tag{5}$$

Here, $\boldsymbol{C}(\boldsymbol{u}) = \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}(\boldsymbol{u}) = 2\boldsymbol{E}(\boldsymbol{u}) + \boldsymbol{I}$ is the right Cauchy-Green deformation tensor. For π, in the present study, we use the Yeoh model given as

$$\begin{aligned}\pi(\boldsymbol{C}(\boldsymbol{u})) &= e_1(i_1(\boldsymbol{u}) - 3) + e_2(i_1(\boldsymbol{u}) - 3)^2\\ &+ e_3(i_1(\boldsymbol{u}) - 3)^3 + \frac{1}{d_1}(i_3(\boldsymbol{u}) - 1)^2\\ &+ \frac{1}{d_2}(i_3(\boldsymbol{u}) - 1)^4 + \frac{1}{d_3}(i_3(\boldsymbol{u}) - 1)^6,\end{aligned}$$

where e_1, e_2, e_3, d_1, d_2 and d_3 denote material parameters, $i_1(\boldsymbol{u})$ and $i_3(\boldsymbol{u})$ denote the first and third invariants defined by

$$\begin{aligned}i_1(\boldsymbol{u}) &= i_3^{-2/3}(\boldsymbol{u})\left(c_1^2(\boldsymbol{u}) + c_2^2(\boldsymbol{u}) + c_3^2(\boldsymbol{u})\right),\\ i_3(\boldsymbol{u}) &= \det\boldsymbol{F}(\boldsymbol{u}),\end{aligned}$$

and $c_1(\boldsymbol{u})$, $c_2(\boldsymbol{u})$, and $c_3(\boldsymbol{u})$ are the principal values of $\boldsymbol{C}(\boldsymbol{u})$.

Using (5) as the constitutive equation, the hyperelastic problem can be defined using the first Piola-Kirchhoff stress tensor defined by

$$\boldsymbol{\Pi}(\boldsymbol{u}) = \boldsymbol{F}(\boldsymbol{u})\boldsymbol{S}(\boldsymbol{u}).$$

In the present study, $\boldsymbol{\nu}$ denotes the outer unit normal on the boundary.

Problem 1 (Hyperelastic problem) *For $\boldsymbol{\phi} \in \mathcal{D}$ and $t \in (0, t_{\mathrm{T}})$, let $\boldsymbol{u}_{\mathrm{D}}(t) : \mathbb{R}^d \to \mathbb{R}^d$ be a given function. Find $\boldsymbol{u}(t) : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ such that*

$$\begin{aligned}-\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}(\boldsymbol{u}(t)) &= \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \ \text{in } \Omega(\boldsymbol{\phi}),\\ \boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u}(t))\boldsymbol{\nu} &= \boldsymbol{0}_{\mathbb{R}^d} \ \text{on } \Gamma_{\mathrm{N}}(\boldsymbol{\phi}),\\ \boldsymbol{u}(t) &= \boldsymbol{u}_{\mathrm{D}}(t) \ \text{on } \Gamma_{\mathrm{D0}}.\end{aligned}$$

If $\boldsymbol{u}_{\mathrm{D}}(t)$ is given appropriately, for the weak solution $\boldsymbol{u}(t)$ to Problem 1, $\tilde{\boldsymbol{u}}(t) = \boldsymbol{u}(t) - \boldsymbol{u}_{\mathrm{D}}(t)$ lies within

$$U = \left\{\boldsymbol{u} \in H^1\left(\mathbb{R}^d;\mathbb{R}^d\right) \mid \boldsymbol{u} = \boldsymbol{0}_{\mathbb{R}^d} \text{ on } \Gamma_{\mathrm{D0}}\right\}, \tag{6}$$

since the domain of $\boldsymbol{u}(t)$ can be extended to $\mathbb{R}^d$ by Calderón's extension theorem. Moreover, in the present paper, we define the admissible set of $\tilde{\boldsymbol{u}}(t)$ by

$$\mathcal{S} = U \cap W^{2,4q}\left(\mathbb{R}^d;\mathbb{R}^d\right) \tag{7}$$

for $q > d$, in order to obtain the domain variation in Y without singular points by the H^1 gradient method [6]. For the simplicity, $\boldsymbol{u}(t)$ is denoted by $\boldsymbol{u}(t)$ or $\boldsymbol{u}$, and $\boldsymbol{u}_{\mathrm{D}}(t)$ is denoted by $\boldsymbol{u}_{\mathrm{D}}$ from here.

For later use, we define the Lagrange function for Problem 1 as

$$\begin{aligned}\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}) &= \int_{\Omega(\boldsymbol{\phi})}\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}(\boldsymbol{u})\boldsymbol{v}\,\mathrm{d}x\\ &+ \int_{\Gamma_{\mathrm{D0}}}\left\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}})\cdot\left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{v})\boldsymbol{\nu}\right) + \boldsymbol{v}\cdot\left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{\nu}\right)\right\}\mathrm{d}\gamma,\end{aligned} \tag{8}$$

where $\boldsymbol{v} \in U$ is introduced as the Lagrange multiplier. Here, the second term on the right-hand side of (8), which is assumed to be zero based on the Dirichlet conditions, was added for use later herein [6]. The first term on the right-hand side of (8) can be rewritten as

$$\begin{aligned}\int_{\Omega(\boldsymbol{\phi})}\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}(\boldsymbol{u})\boldsymbol{v}\,\mathrm{d}x &= \int_{\Omega(\boldsymbol{\phi})}\{\boldsymbol{\nabla}\cdot(\boldsymbol{\Pi}(\boldsymbol{u})\boldsymbol{v})\\ &- \boldsymbol{\Pi}(\boldsymbol{u})\cdot\left(\boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}}\right)^{\mathrm{T}}\}\mathrm{d}x\\ &= \int_{\partial\Omega(\boldsymbol{\phi})}(\boldsymbol{\Pi}(\boldsymbol{u})\boldsymbol{v})\cdot\boldsymbol{\nu}\,\mathrm{d}\gamma - \int_{\Omega(\boldsymbol{\phi})}\boldsymbol{\Pi}(\boldsymbol{u})\cdot\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\,\mathrm{d}x,\end{aligned}$$

where $\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}] = \partial\boldsymbol{v}/\partial\boldsymbol{x}^{\mathrm{T}}$, and "$\cdot$" denotes the scalar product. Moreover, considering $\boldsymbol{S}(\boldsymbol{u}) = \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{u})$,

$$\begin{aligned}-\int_{\Omega(\boldsymbol{\phi})}\boldsymbol{\Pi}(\boldsymbol{u})\cdot\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\,\mathrm{d}x &= -\int_{\Omega(\boldsymbol{\phi})}(\boldsymbol{F}(\boldsymbol{u})\boldsymbol{S}(\boldsymbol{u}))\cdot\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\,\mathrm{d}x\\ &= -\int_{\Omega(\boldsymbol{\phi})}\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}]\,\mathrm{d}x\end{aligned} \tag{9}$$

holds, where

$$\begin{aligned}\boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}] &= \frac{1}{2}\left\{\boldsymbol{F}'^{\mathrm{T}}(\boldsymbol{u})[\boldsymbol{v}]\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{F}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\right\}\\ &= \boldsymbol{E}_{\mathrm{L}}(\boldsymbol{v}) + \boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{u},\boldsymbol{v}) = \left(e'_{ij}(\boldsymbol{u})[\boldsymbol{v}]\right)_{ij}.\end{aligned}$$

Then, using (9), (8) can be rewritten as

$$\begin{aligned}\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}) &= -\int_{\Omega(\boldsymbol{\phi})}\boldsymbol{S}(\boldsymbol{u})\cdot\boldsymbol{E}'(\boldsymbol{u})[\boldsymbol{v}]\,\mathrm{d}x\\ &+ \int_{\Gamma_{\mathrm{D0}}}\{(\boldsymbol{u} - \boldsymbol{u}_{\mathrm{D}})\cdot\left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{v})\boldsymbol{\nu}\right) + \boldsymbol{v}\cdot\left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{\nu}\right)\}\mathrm{d}\gamma.\end{aligned} \tag{10}$$

If $\boldsymbol{u}$ is the solution to Problem 1,

$$\mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi},\boldsymbol{u},\boldsymbol{v}) = 0 \tag{11}$$

holds for all $\boldsymbol{v} \in U$. Then, (11) agrees with the weak form of Problem 1.

4. Shape optimization problem

Using $\boldsymbol{u}$, we define the shape optimization problem as follows. Let $\alpha_1, \cdots, \alpha_m$ be the constants denoting the desired value of $\boldsymbol{u}_{\mathrm{D}} \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u})\boldsymbol{\nu}\right)$ at $t_1, \cdots, t_m \in (0, t_{\mathrm{T}}]$, respectively. In the present paper, we defined

$$f_0(\boldsymbol{\phi}, \boldsymbol{u}) = \sum_{i\in\{1,\cdots,m\}} f_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i)) \tag{12}$$

as the objective cost function, where

$$f_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i)) = \int_{\Gamma_{\mathrm{D}0}} \left|\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u}(t_i))\boldsymbol{\nu}\right) - \alpha_i\right|^2 \mathrm{d}\gamma.$$

Moreover, we define

$$f_1(\boldsymbol{\phi}) = \int_{\Omega(\boldsymbol{\phi})} \mathrm{d}x - c_1 \tag{13}$$

as a constraint cost function, where c_1 is a positive constant for which there exists $\boldsymbol{\phi} \in \mathcal{D}$ such that $f_1(\boldsymbol{\phi}) \leq 0$.

Using these cost functions, we construct the following shape optimization problem.

Problem 2 (Squared error norm minimization) *Let $f_0(\boldsymbol{\phi}, \boldsymbol{u})$ and $f_1(\boldsymbol{\phi})$ be defined as in (12) and (13), respectively. Find $\boldsymbol{\phi}$ such that*

$$\min_{\boldsymbol{\phi}\in\mathcal{D}}\Big\{f_0(\boldsymbol{\phi}, \boldsymbol{u}) \mid f_1(\boldsymbol{\phi}) \leq 0, \\ \boldsymbol{u}(t) \in \mathcal{S},\ t \in (0, t_{\mathrm{T}}),\ \textit{Problem 1}\Big\}.$$

5. Shape derivative of the cost functions

In order to solve Problem 2 by the gradient method, the Fréchet derivatives of f_0 and f_1 with respect to domain variation, which we refer to as the shape derivative, are required. Let $\boldsymbol{\varphi} \in X$ be the domain variation from $\boldsymbol{\phi}$. If there exist $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ such that $f_0'(\boldsymbol{\phi}, \boldsymbol{u})[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_0, \boldsymbol{\varphi}\rangle$ and $f_1'(\boldsymbol{\phi})[\boldsymbol{\varphi}] = \langle \boldsymbol{g}_1, \boldsymbol{\varphi}\rangle$ for all $\boldsymbol{\varphi} \in X$, we refer to $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ as the shape derivatives of f_0 and f_1, respectively. Here, $\langle \cdot, \cdot \rangle$ denotes the dual product.

Since f_0 is a functional of $\boldsymbol{u}$, $\boldsymbol{g}_0$ is obtained as follows using the Lagrange multiplier method. We define

$$\mathscr{L}_0(\boldsymbol{\phi}, \boldsymbol{u}, \boldsymbol{v}_{01}, \cdots, \boldsymbol{v}_{0m}) = \sum_{i\in\{1,\cdots,m\}} \mathscr{L}_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i}) \\ = \sum_{i\in\{1,\cdots,m\}} \left(f_{0i}(\boldsymbol{\phi}, \boldsymbol{u}(t_i)) + \mathscr{L}_{\mathrm{M}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})\right)$$

as the Lagrangian for f_0, where $\boldsymbol{v}_{0i}$ is introduced as the Lagrange multipliers for Problem 1 at $t = t_i$ such that $\tilde{\boldsymbol{v}}_{0i} = \boldsymbol{v}_{0i} + 2\left(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u}(t_i))\boldsymbol{\nu}\right) - \alpha_i\right)\boldsymbol{u}_{\mathrm{D}}(t_i) \in U$. The shape derivative of $\mathscr{L}_{0i}$ can be written as

$$\mathscr{L}_{0i}'(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{\varphi}, \boldsymbol{u}^*(t_i), \boldsymbol{v}_{0i}^*] \\ = \mathscr{L}_{0i\boldsymbol{\phi}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{\varphi}] \\ + \mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] \\ + \mathscr{L}_{0i\boldsymbol{v}_{0i}}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{v}_{0i}^*], \tag{14}$$

where $\boldsymbol{u}^*(t_i) \in U$ and $\boldsymbol{v}_{0i}^* \in U$ are the partial shape derivatives of $\boldsymbol{u}(t_i)$ and $\boldsymbol{v}_{0i}$, respectively [6].

Here, if $\boldsymbol{u}$ is the solution of Problem 1, the third term on the right-hand side of (14) becomes 0. The second term on the right-hand side of (14) becomes

$$\mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] \\ = -\int_{\Omega(\boldsymbol{\phi})} \big(\boldsymbol{S}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)] \cdot \boldsymbol{E}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \\ + \boldsymbol{S}(\boldsymbol{u}(t_i)) \cdot \boldsymbol{E}''(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}, \boldsymbol{u}^*(t_i)]\big)\mathrm{d}x \\ + \int_{\Gamma_{\mathrm{D}0}} \Big[\boldsymbol{u}^*(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{v}_{0i})\boldsymbol{\nu}\right) + \big\{\boldsymbol{v}_{0i} \\ + 2\left(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u}(t_i))\boldsymbol{\nu}\right) - \alpha_i\right)\boldsymbol{u}_{\mathrm{D}}(t_i)\big\} \\ \cdot \left(\boldsymbol{\Pi}'^{\mathrm{T}}(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\boldsymbol{\nu}\right)\Big]\mathrm{d}\gamma, \tag{15}$$

where

$$\boldsymbol{S}'(\boldsymbol{u})[\boldsymbol{v}] = \sum_{(i,j)\in\{1,\ldots,d\}^2} \frac{\partial \boldsymbol{S}(\boldsymbol{u})}{\partial e_{ij}(\boldsymbol{u})} e_{ij}'(\boldsymbol{u})[\boldsymbol{v}],$$

$$\boldsymbol{E}''(\boldsymbol{u})[\boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{E}_{\mathrm{BL}}(\boldsymbol{v}, \boldsymbol{w}),$$

$$\boldsymbol{\Pi}'(\boldsymbol{u})[\boldsymbol{v}] = \boldsymbol{F}'(\boldsymbol{u})[\boldsymbol{v}]\boldsymbol{S}(\boldsymbol{u}) + \boldsymbol{F}(\boldsymbol{u})\boldsymbol{S}'(\boldsymbol{u})[\boldsymbol{v}].$$

If we use the same relation used in (9), (15) becomes

$$-\int_{\Omega(\boldsymbol{\phi})} \big(\boldsymbol{S}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)] \cdot \boldsymbol{E}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \\ + \boldsymbol{S}(\boldsymbol{u}(t_i)) \cdot \boldsymbol{E}''(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}, \boldsymbol{u}^*(t_i)]\big)\mathrm{d}x \\ = -\int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \cdot \boldsymbol{F}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\,\mathrm{d}x.$$

Moreover, assuming the relations $\boldsymbol{u}^*(t_i) = \boldsymbol{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{D}0}$ and $\boldsymbol{\Pi}'^{\mathrm{T}}(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{\nu} = \boldsymbol{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{N}}(\boldsymbol{\phi})$, we have

$$\int_{\partial\Omega(\boldsymbol{\phi})} \left(\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)\right) \cdot \boldsymbol{\nu}\ \mathrm{d}\gamma \\ - \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] \cdot \boldsymbol{F}'(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\,\mathrm{d}x \\ = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)\ \mathrm{d}x.$$

From the above relations, (15) can be rewritten as

$$\mathscr{L}_{0i\boldsymbol{u}(t_i)}(\boldsymbol{\phi}, \boldsymbol{u}(t_i), \boldsymbol{v}_{0i})[\boldsymbol{u}^*(t_i)] \\ = \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}]\boldsymbol{u}^*(t_i)\ \mathrm{d}x \\ + \int_{\Gamma_{\mathrm{D}0}} \Big[\boldsymbol{u}^*(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{v}_{0i})\boldsymbol{\nu}\right) + \big\{\boldsymbol{v}_{0i} \\ + 2\left(\boldsymbol{u}_{\mathrm{D}}(t_i) \cdot \left(\boldsymbol{\Pi}^{\mathrm{T}}(\boldsymbol{u}(t_i))\boldsymbol{\nu}\right) - \alpha_i\right)\boldsymbol{u}_{\mathrm{D}}(t_i)\big\} \\ \cdot \left(\boldsymbol{\Pi}'^{\mathrm{T}}(\boldsymbol{u}(t_i))[\boldsymbol{u}^*(t_i)]\boldsymbol{\nu}\right)\Big]\mathrm{d}\gamma$$

for all $\boldsymbol{u}^*(t_i)$ such that $\boldsymbol{u}^*(t_i) = \boldsymbol{0}_{\mathbb{R}^d}$ on $\Gamma_{\mathrm{D}0}$. Then, (15) becomes 0 if $\boldsymbol{v}_{0i}$ is the solution of the following adjoint problem.

Problem 3 (Adjoint problem for f_0) *Let $\boldsymbol{u}(t_i)$ be the solution of Problem 1. Find $\boldsymbol{v}_{0i} : \Omega(\boldsymbol{\phi}) \to \mathbb{R}^d$ such that*

$$-\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{\Pi}'(\boldsymbol{u}(t_i))[\boldsymbol{v}_{0i}] = \boldsymbol{0}_{\mathbb{R}^d}^{\mathrm{T}} \ \text{in}\ \Omega(\boldsymbol{\phi}),$$

$$\mathbf{\Pi}'^{\mathrm{T}}\left(\boldsymbol{u}\left(t_i\right)\right)\left[\boldsymbol{v}_{0i}\right]\boldsymbol{\nu} = \mathbf{0}_{\mathbb{R}^d} \ \ on\ \Gamma_{\mathrm{N}}\left(\boldsymbol{\phi}\right),$$

$$\boldsymbol{v}_{0i} = -2\left(\boldsymbol{u}_{\mathrm{D}}\left(t_i\right)\cdot\left(\mathbf{\Pi}^{\mathrm{T}}\left(\boldsymbol{u}\left(t_i\right)\right)\boldsymbol{\nu}\right) - \alpha_i\right)\boldsymbol{u}_{\mathrm{D}}\left(t_i\right) \ \ on\ \Gamma_{\mathrm{D0}}.$$

In order to obtain the domain variation in Y without singular points by the H^1 gradient method, $\tilde{\boldsymbol{v}}_{0i} \in \mathcal{S}$ is required [6].

Let $\boldsymbol{u}\left(t_i\right)$ and $\boldsymbol{v}_{0i}$ be solutions of Problem 1 and Problem 3, respectively. Then, (14) becomes

$$\mathscr{L}_{0i\boldsymbol{\phi}}\left(\boldsymbol{\phi},\boldsymbol{u}\left(t_i\right),\boldsymbol{v}_{0i}\right)\left[\boldsymbol{\varphi}\right] = f'_{0i}\left(\boldsymbol{\phi},\boldsymbol{u}\right)\left[\boldsymbol{\varphi}\right]$$
$$= \int_{\Gamma_{\mathrm{N}}(\boldsymbol{\phi})} \boldsymbol{g}_{0i\mathrm{N}}\cdot\boldsymbol{\varphi}\ \mathrm{d}\gamma = \left\langle \boldsymbol{g}_{0i},\boldsymbol{\varphi}\right\rangle, \tag{16}$$

where

$$\boldsymbol{g}_{0i\mathrm{N}} = -\boldsymbol{S}\left(\boldsymbol{u}\left(t_i\right)\right)\cdot\boldsymbol{E}'\left(\boldsymbol{u}\left(t_i\right)\right)\left[\boldsymbol{v}_{0i}\right]\boldsymbol{\nu}.$$

For f_0, we have

$$f'_0\left(\boldsymbol{\phi},\boldsymbol{u}\right)\left[\boldsymbol{\varphi}\right] = \sum_{i\in\{1,\cdots,m\}}\left\langle \boldsymbol{g}_{0i},\boldsymbol{\varphi}\right\rangle = \left\langle \boldsymbol{g}_{0},\boldsymbol{\varphi}\right\rangle. \tag{17}$$

Moreover, for the shape derivative of f_1, we have

$$f'_1\left(\boldsymbol{\phi}\right)\left[\boldsymbol{\varphi}\right] = \int_{\Gamma_{\mathrm{N}}(\boldsymbol{\phi})} \boldsymbol{\nu}\cdot\boldsymbol{\varphi}\ \mathrm{d}\gamma = \left\langle \boldsymbol{g}_{1},\boldsymbol{\varphi}\right\rangle. \tag{18}$$

6. Solution

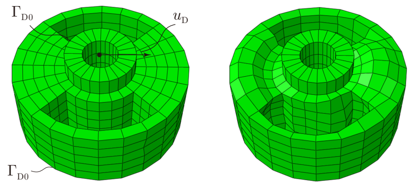
The algorithm for solving Problem 2 can be shown based on the sequential quadratic programming [6]. In this algorithm, the H^1 gradient method is used for reshaping with shape derivatives $\boldsymbol{g}_0$ and $\boldsymbol{g}_1$ in (17) and (18), respectively.

7. Numerical example

We developed a computer program to solve Problem 2. In the program, a commercial finite element program, Abaqus 6.9 (Dassault Systèmes), is used to solve Problem 1 and Problem 3. Moreover, OPTISHAPE-TS 2011 (Quint Corporation) is used to solve the boundary value problem in the H^1 gradient method.

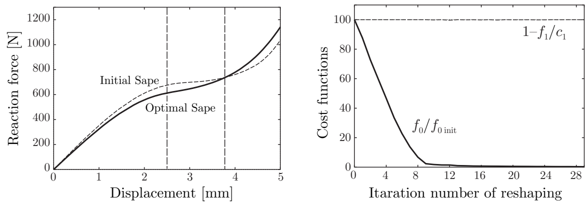
Figure 1(a) shows a finite element model of the rubber bushing used as an example. The diameter of the outer cylinder is 50.0 [mm]. The outer and inner cylinders are assumed to be the homogeneous and non-homogeneous Dirichlet boundaries, respectively. The nodes on the inner cylinder are connected with a rigid element. The arrow of $\boldsymbol{u}_{\mathrm{D}}$ shows the compulsory displacement of the rigid element, the magnitude of which is 5.0 [mm]. For f_0, we assume that $m = 3$, and $\{\|\boldsymbol{u}_{\mathrm{D}}(t_1)\|, \|\boldsymbol{u}_{\mathrm{D}}(t_2)\|, \|\boldsymbol{u}_{\mathrm{D}}(t_3)\|\} = \{2.5, 3.75, 5.0\}$ [mm]. For α_1, α_2 and α_3, we use a 10% decrease, no change, and a 10% increase for the values of $\boldsymbol{u}_{\mathrm{D}}\cdot\left(\mathbf{\Pi}^{\mathrm{T}}\left(\boldsymbol{u}\right)\boldsymbol{\nu}\right)$ at $t = t_1$, t_2, and t_3, respectively.

Figure 1(b) shows the optimum shape obtained by the developed program. The reaction force of the rigid element defined by $\|\int_{\Gamma_{\mathrm{D0}}}\mathbf{\Pi}^{\mathrm{T}}\left(\boldsymbol{u}\left(t\right)\right)\boldsymbol{\nu}\ \mathrm{d}\gamma\|$ with respect to compulsory displacement $\|\boldsymbol{u}_{\mathrm{D}}(t)\|$ is shown in Fig. 2(a). Figure 2(b) shows the iteration histories of the cost functions with respect to the number of reshapings, where $f_{0\,\mathrm{init}}$ and c_1 denote the values of f_0 and the volume, respectively, for the initial shape.



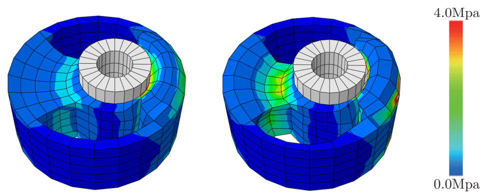
(a) Initial (b) Optimized

Fig. 1. Finite element models of simple rubber bushings



(a) Reaction force functions (b) Iteration histories

Fig. 2. Graphs for shape optimization analysis



(a) Initial (b) Optimized

Fig. 3. Initial and optimized von Mises stresses

Based on these results, f_0 decreases monotonically under the constraint of f_1 (Fig. 2(b)), and the desired reaction force function is obtained (Fig. 2(a)).

In addition, Fig. 3 shows the distributions of the von Mises stress of the initial and optimum shapes at $t = t_3$. The results confirm that as the result of increasing the reaction force at $t = t_3$, the von Mises stress in the optimum shape increases.

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