Construction of Finite-Element Models Conforming to Prescribed Boundary Shapes

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This paper presents two techniques, called the morphing method and the fitting method, for deforming finite-element models to conform to prescribed boundary shapes of domains. The morphing method involves selecting nodes on a finite-element model and moving them to desired locations under suitable constraints in reference to specified image data. The fitting method involves deforming a finite-element model to fit into a prescribed domain that can be given with pixel or voxel image data. These methods have been derived as solutions to shape optimization problems for morphing and fitting using the traction method. The morphing method can be applied to construct a finite-element model of the spine to match that of patients with idiopathic scoliosis by referring to X-ray photographs. The fitting method can be applied to construct a skeletal part that matches CT image data.

Key Words: Optimum Design, Finite Element Method, Shape Optimization, Traction Method, Morphing Method

1. Introduction

Computational mechanics has been applied not only in industry, but also in the field of medicine. One of the biggest difficulties in applying computational methods such as the finite-element method to the human body is how to construct numerical models from medical image data, such as X-ray photographs and pixel or voxel image data obtained by computer tomography (CT) or magnetic resonance imaging (MRI).

Recently, some methods for constructing finite-element models of skeletal parts from CT image data have been developed. One simple method is to convert voxel image data to cubic finite-elements directly. Another method is to use the technique of contour extraction from cross-sectional image data and automatic generation of triangular surface patches and tetrahedral meshes, which has been applied to the construction of finite-element models of the mandible, femur and lumbar vertebrae. A method based on transformation of existing finite-element models to conform to triangular surface patches has been presented using the deformation induced by the uniform traction and contact conditions of the surface patches. An application of the basis vector method to the construction of finite-element models of skeletal parts has also been reported.

However, some difficulties remain in the generation of hexahedral structured meshes and in their applicability to large-scale models, such as for the spinal system. Validated finite-element meshes of the spine are essential in conducting buckling analyses to investigate the etiology of idiopathic scoliosis in patients and to analyze effective reinforcement parts for treatment based on the optimization theory. The methods used heretofore have certain drawbacks such as they require high-volume CT image data to construct large-scale models for use in creating fine finite-element models, can not generate hexahedral structured meshes, require the treatment of contact conditions, or have limited degrees of freedom for shape variation.

One feasible way to overcome such difficulties is to deform validated finite-element models to conform to the available medical image data without having to deal with contact conditions and limitations on degrees of freedom for shape variation. This paper presents two types of deforming techniques, called the morphing method and the fitting method, which are based on shape optimization problems.

The morphing method involves selecting nodes on a finite-element model and moving them to desired locations under suitable constraints in reference to the relevant image data. This method can be applied to construct finite-element models of the spine of patients with idiopathic scoliosis by using medical image data, such as X-ray photographs and pixel or voxel image data obtained by CT or MRI.

The fitting method involves deforming a finite-element model to fit into a prescribed domain that can be described with pixel...
or voxel image data. This method can be applied to construct models of skeletal parts that agree with CT image data.

These methods have been derived as solutions to shape optimization problems of domains that cause morphing and fitting deformations. For morphing deformations, the squared distance between the selected nodes and the desired locations and the domain measure are used as an objective functional and a constraint functional, respectively. An integral of a signed distance function is chosen as an objective functional for fitting deformations. Shape gradients for these shape optimization problems can be obtained by the Lagrange multiplier method and the formula of material derivative.

Smooth reshaping using the shape gradients can be accomplished by the traction method using the Robin condition that was proposed as an improved version of the traction method using the Neumann condition.

A significant feature of the morphing method is that it facilitates the construction of finite-element models from incomplete data such as X-ray photographs or partial CT image data. The fitting method is characterized by its applicability to validated coarse finite-element meshes, flexibility in shape variation in comparison with the basis vector method, and simplicity of the algorithm in comparison with the method using uniform traction and contact conditions.

2. Morphing method

Consider that we select certain nodes on a finite-element model and move them to desired locations under suitable constraints in reference to X-ray photographs or CT image data. The key to moving the nodes successfully is to maintain boundary smoothness.

2.1 Morphing problem

Let us consider that a linear elastic continuum defined in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$), having the boundary $\Gamma$, is expected to move to a bounded reference domain $\Omega^\text{ref} \subset \mathbb{R}^n$, and that specified points in particular $x_k \in \Omega$ ($k = 1, 2, \ldots, q$) move to target positions $x_k^\text{ref} \in \Omega^\text{ref}$ ($k = 1, 2, \ldots, q$) respectively as shown in Fig. 1. This domain moving problem involving a domain measure constraint can be formulated as

$$\min_{\Omega \subset \mathbb{R}^n} \int_{\Omega} \sum_{k=1}^q \delta(x - x_k) \| x_k^\text{ref} - x \|^2 \, dx$$

where $\delta(\cdot)$ denotes the Dirac delta function.

The shape gradient of this problem can be obtained by the Lagrange multiplier method. Using $\Lambda$ as a Lagrange multiplier for the domain measure constraint, the Lagrange functional $L$ and its derivative $\dot{L}$ with respect to domain variation are obtained using the formula of material derivative as

$$L = \int_{\Omega} \sum_{k=1}^q \delta(x - x_k) \| x_k^\text{ref} - x \|^2 \, dx + \Lambda \left( \int_{\Omega} dx - \int_{\Omega^\text{ref}} dx \right)$$

$$\dot{L} = \int_{\Omega} \sum_{k=1}^q \delta(x - x_k) (x_k^\text{ref} - x) \cdot \dot{x} \, dx$$

$$= -2 \int_{\Omega} \sum_{k=1}^q \delta(x - x_k) (x_k^\text{ref} - x) \cdot \dot{x} dx + \dot{\Lambda} \left( \int_{\Omega} dx - \int_{\Omega^\text{ref}} dx \right)$$
where \(\cdot\), \(v\) and \(V\) denote the material derivative, the outward unit normal vector and the velocity of the domain variation, respectively. In Eq.(4), the relations \(\delta(x - x_k) = 0\) and \(\dot{x} = V\) were used.

Using \(A\) determined to satisfy the domain measure constraint, the derivative of the Lagrange functional is obtained as

\[
L_\lambda = \int_\Omega G_0 \cdot V \, dx + \int_\Gamma G_1 v \cdot V \, d\Gamma
\equiv (G_0, V) + \lambda (G_1 v, V)
\]

\[
G_0 = -2 \sum_{i=1}^q \delta(x - x_k) (x_k^\text{ref} - x), \quad G_1 \equiv 1
\]

where \(G_0\) and \(G_1 v\) are the shape gradients for the objective functional and the domain measure constraint in this problem, respectively. In this paper, the sign \(\equiv\) is used for definitions.

### 2.2 Morphing method using the traction method

Let us consider reshaping with the shape gradients \(G_0\) and \(G_1 v\) obtained above. Because \(G_0\) consists of Dirac delta functions, there was concern that the shape gradients might lack smoothness even if the traction method was applied. Let us consider a simple morphing problem in which a point on a beam-like finite-element model was expected to move to the target point shown in Fig. 2(a). A sharply pointed shape was obtained with the simple iteration of the improved traction method using the Robin condition(8) as shown in Fig. 2(b).

Recalling that the traction method works as a boundary smoother during reshaping,(11) it can be expected that repetition of the traction method will improve boundary smoothness. The procedure of the proposed morphing method is to iterate the following steps using \(\Delta s\) as a step size parameter for reshaping at one iteration and \(\alpha > 0\) as the boundary stiffness parameter:(8)

1. Solve \(V^{[1]}_0, V_1 \in D:\)

\[
a(V^{[1]}_0, y) + \alpha(\langle V^{[1]}_0 \cdot v \rangle_v, y) = -(G_0, y) \quad \forall y \in D. \quad (7)
\]

\[
a(V_1, y) + \alpha(\langle V_1 \cdot v \rangle_v, y) = -(G_1 v, y) \quad \forall y \in D. \quad (8)
\]
(ii) Set \( m = 2 \) and substitute the trace of \( V_0^{[m-1]} \) on \( \Gamma \) into \( G^{[m]} \) and solve \( V_0^{[m]} \in D: \)

\[
a(V_0^{[m]}, y) + \partial((V_0^{[m]} \cdot \nu) V_0^{[m]}) = (G^{[m]}, y) \quad \forall y \in D. \tag{9}
\]

Substitute \( m+1 \) for \( m \) and iterate this step until satisfactory smoothness is obtained.

(iii) Calculate the Lagrange multiplier \( \Lambda \) with the following equation and reshape with \( I + \Delta \lambda \{V_0^{[m]} + \Lambda V_1\} \), where \( I \) denotes the identity mapping.

\[
\Lambda = \frac{(G V_0, \Delta \lambda V_0^{[m]}) + \left( \int_{\Omega} dx - \int_{\Omega_1} dx \right)}{(G V_0, \Delta \lambda V_0^{[m]})}. \tag{10}
\]

Return to the Step (i) until the objective functional becomes satisfactory small.

The bilinear forms \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) are defined in Eq. (5). The bilinear form \( a(\cdot, \cdot) \) is defined as

\[
a(V, y) = \int_{\Omega} C_{ijkl} V_{ij} y_{kl} \, dx \tag{11}
\]

where \( C_{ijkl} \) is the stiffness of the linear elastic continuum. In Eq. (11), the Einstein summation convention and the gradient notation \( \partial(\cdot) = \partial(\cdot) / \partial x \) were used. The set \( D \) is defined as

\[
D = \{ V \in (H^1(\Omega))^n \mid \text{shape constraints} \}. \tag{12}
\]

For example, Eq. (7) means determining velocity \( V_0^{[1]} \) by solving a displacement of a pseudo-elastic body defined in domain \( \Omega \) accompanied with a distributed spring of coefficient \( \alpha \) connecting the boundary \( \Gamma \) of the pseudo-elastic body to settled points in the normal direction by the loading of a pseudo-external force \( G_0 \) in domain \( \Omega \) under constraints on displacement for shape variation. These equations can be also implemented through the use of the finite element method by adding the stiffness matrices of the distributed spring for each boundary element to the global stiffness matrix.

When the repetitive traction method was actually applied to the morphing problem shown in Fig. 2(a), a smooth boundary was obtained as shown in Fig. 2(c).

2.3 Numerical Example One of the most effective applications of the morphing method is to construct a finite-element model of the spine of a patient with idiopathic scoliosis. Figure 3(a) shows an X-ray photograph of a patient whose spine the finite-element model is expected to match. In this morphing example, it was assumed that only one point on the surface at the eighth thoracic vertebra would be moved under the conditions of a restrained bottom surface and top point of the model.

Figures 3(c) and (d) illustrate the results of the iteration with the traction method and the repetitive traction method, respectively. Iteration histories with the traction method and the repetitive traction method are shown in Figs. 3(e) and (f), respectively. More precise agreement can be obtained by using additional moving conditions.

3. Fitting method

Let us assume that a finite-element model is positioned near the desired location in the example for the morphing method and that the model deforms to fit the pixel or voxel image data obtained by CT or MRI.

3.1 Fitting problem Let us consider that a linear elastic continuum defined in a bounded domain \( \Omega \subset \mathbb{R}^n (n = 2, 3) \), having the boundary \( \Gamma \) that is included in the domain \( \Omega_{\text{ext}} \subset \mathbb{R}^n \) for pixel or voxel image data, is expected to move to a bounded reference domain \( \Omega_{\text{ref}} \subset \mathbb{R}^n \), having the boundary \( \Gamma_{\text{ref}} \), as shown in Fig. 4.

To formulate a fitting problem, let us introduce a signed distance function \( \phi(x) : \Omega_{\text{ext}} \ni \phi \in \mathbb{R} \) defined as

\[
\phi(x) = \begin{cases} 
1 & \text{if } x \in \Omega_{\text{ref}} \\
-1 & \text{if } x \notin \Omega_{\text{ref}} 
\end{cases} \tag{13}
\]

\[
\chi_{\text{ext}}(x) = \begin{cases} 
1 & \text{if } x \in \Omega_{\text{ext}} \\
0 & \text{if } x \notin \Omega_{\text{ext}} 
\end{cases} \tag{14}
\]

Fig.5 Allocation of distance for pixel data
where \( d(x) : \Omega^{\text{init}} \ni x \mapsto d \in \mathbb{R} \) denotes the distance at \( x \in \Omega^{\text{init}} \) from the boundary \( \Gamma^{\text{ref}} \) as shown in Fig. 4. Using \( \phi(x) \), a fitting problem can be formulated as

\[
\min_{\Omega \subset \mathbb{R}^n} \left\{ J = \int_{\Omega} \phi(x) \, dx \right\}.
\]

(15)

It was actually confirmed that the objective functional \( J \) became minimum when \( \Omega \) agreed with \( \Omega^{\text{ref}} \).

Using the formulation of material derivative, the derivative of the objective functional \( J \) with respect to domain variation is obtained as

\[
\dot{J} = \int_{\Gamma} \phi(x) \, V \cdot d\Gamma = \langle G \nu, V \rangle,
\]

where \( G \nu \) is the shape gradient for the objective functional in this problem.

### 3.2 Fitting method using the traction method

When the target domain \( \Omega^{\text{ref}} \) consists of pixel or voxel image data, it will not be effective to evaluate the signed distance in the strict sense of the definition. Considering that the traction method works as a boundary smoother during reshaping, we can evaluate the signed distance as a constant in each pixel or voxel.

The procedure of the proposed fitting method is to apply the improved traction method repeatedly using the shape gradient \( G \nu \) of the fitting problem as follows:}

\[
\dot{J} = \int_{\Gamma} \phi(x) \, V \cdot d\Gamma \equiv \langle G \nu, V \rangle,
\]

\( G \equiv \phi(x) \)
(i) Extract a contour from each cross section of the pixel or voxel image data and set \( d = 0 \) for the pixels or voxels on the contour.

(ii) Allocate a distance unit to \( d \) for the pixels or voxels in contact with the pixels or voxels on the contour and twice the distance unit \( d \) for the pixels or voxels in contact with the pixels or voxels contiguous with the distance unit, and iterate in a similar fashion for all the pixels or voxels as shown in Fig. 5.

(iii) Iterate to solve \( V \in D: \)

\[
a(V, y) + a((V \cdot y)v, y) = \langle \chi_{D^{int}}(x)d(x), y \rangle \forall y \in D \quad (17)
\]

and to reshape with \( I + \Delta sV \) using a step size parameter \( \Delta s \) until the objective functional converges.

3.3 Numerical Example Using the fitting method, a finite-element model of the third lumber vertebra was fitted to CT image data as shown in Fig. 6. In this analysis, the domain of the finite-element model consisting of cortical bone was chosen as \( \Omega \). Therefore, the finite elements consisting of cancerous bone were followed by the deformation of the finite elements of cortical bone.

4. Conclusion

This paper has presented two types of methods, called the morphing method and the fitting method, for deforming finite-element models.

The morphing method was proposed as the solution of a shape optimization problem involving minimization of the squared distance between the specified nodes of the finite-element model and the target positions. Although the shape gradient of the problem consisted of Dirac delta functions that lacked sufficient smoothness, repetitive use of the traction method resolved the insufficiency.

The fitting method was proposed as the solution of a shape optimization problem involving minimization of the integral of the signed distance that was defined as the distance from the boundary of the target domain, using the positive and negative signs for the outside and the inside of the target domain, respectively. The shape gradient of the problem consisted of the signed distance.

Considering that the target domain consisted of pixel or voxel image data and that the traction method works as a boundary smoother during reshaping, it was proposed that the signed distance be evaluated as a constant in each pixel or voxel and that reshaping be done by the traction method.

To confirm the validity of the two methods, the morphing of a finite-element model of the spine to conform to that of a patient with idiopathic scoliosis and the fitting of a finite-element model of the third lumber vertebra to CT image data were demonstrated.

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References


