Solution to Shape Identification Problem of Unsteady Heat-Conduction Fields

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This paper presents a numerical analysis method for shape determination problems of unsteady heat-conduction fields in which time histories of temperature distributions on prescribed subboundaries or time histories of gradient distributions of temperature in prescribed subdomains have prescribed distributions. The square error integrals between the actual distributions and the prescribed distributions on the prescribed subboundaries or in the prescribed subdomains during the specified period of time are used as objective functionals. Reshaping is accomplished by the traction method that was proposed as a solution to shape optimization problems of domains in which boundary value problems are defined. The shape gradient functions of these shape determination problems are derived theoretically using the Lagrange multiplier method and the formulation of material derivative. The time histories of temperature distributions are evaluated using the finite-element method for a space integral and the Crank–Nicolson method for a time integral. Numerical analyses of nozzle and coolant flow passage in a wing are demonstrated to confirm the validity of this method. © 2003 Wiley Periodicals, Inc. Heat Trans Asian Res, 32(3): 212–226, 2003; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/htj.10086

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1. Introduction

Cases in which the heat transfer phenomenon influences the performance of machines and structures are abundant. For example, the technique of applying a special clay to a sword blade in varying amounts of thickness depending on the position on the blade ensured the control of the heat transfer phenomenon in the hardening process of the traditional Japanese sword. The method is a very rational one devised in order to produce a metal form which has the appropriate structure at any given position as a result of controlling the thickness of the special clay at each position. Based on modern engineering which considers mathematical models, it is possible to consider a numerical solution to the problem which determines the boundary shape of the special clay so that this technique may include the temperature history for the heat transfer field produced at the boundary where the special clay came in contact with the sword blade, for each point of the boundary. In this paper, a new practical

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method of numerical analysis for the shape identification problem of unsteady heat-conduction fields to control temperature distribution or temperature gradient distribution is presented. The shape identification problem in this unsteady heat-conduction field is an inverse problem.

There has been much research on the numerical analysis of inverse problems for unsteady heat-conduction fields focused on the identification of time histories of temperature and heat flux [1, 2], the intensity and position of the heat source [3, 4], and characteristic values such as the heat conductivity coefficient and the heat transfer coefficient [5]. However, the research on boundary shape identification has been far from satisfactory. A study limiting the number of design variables to two or three has been made [6, 7]. Even for the steady-state problem, there have not been many research reports [8–12]. In previous studies of numerical analysis for shape identification, the numerical optimization procedure involves first providing a substitute model with a finite number of degrees of freedom as a continuum model and then formulating the optimization problems. Based on this model, a shape optimization (or shape identification) problem, in which the design variables are defined in a finite dimensional vector space, can be analyzed numerically using mathematical programming techniques. This method, however, was not effective in locating an optimal solution to problems with a large number of design variables because of the large number of dimensions in the design space. Thus, the study of the boundary shape identification has not been active.

On the other hand, the present authors have focused on the solution of the shape determination problem based on the distributed sensitivity function which uses adjoint variables. In previous papers, we presented a numerical analysis method for these shape inverse problems of steady heat-conduction fields and potential flow fields in which the distributions of state functions (temperature [13], velocity [14], and pressure [15]) on subboundaries or in subdomains were controlled to the prescribed distributions. Reshaping was accomplished by the traction method [16, 17] which was proposed by one of the authors as a solution to shape optimization problems in which boundary value problems were defined.

In this study, we applied the traction method to the prescribed problem of time histories of temperature distributions or time histories of temperature gradient distributions on unsteady heat-conduction fields. The traction method is a method of applying the gradient method of the distribution system which directly uses the sensitivity functions (shape gradient functions) of the domain variation which are theoretically derived from the optimization problem. In the traction method, the domain variations that minimize the objective functions are obtained as solutions of pseudolinear elastic problems of continua defined on the design domains and loaded with pseudodistributed traction in proportion to the shape gradient functions on the design domains. The numerical solutions of both the shape gradient function and the pseudolinear elastic problems, used for evaluation of the domain variation, can be obtained using the finite-element method or the boundary-element method. Therefore, the traction method can be applied to complex shape determination problems with a large number of design variables. Additionally, this method is advantageous in that the oscillating phenomenon does not occur during the iterative process of shape optimization because the design boundaries are not moved by the movement of finite-element nodal points but by the traction in proportion to the shape gradient functions [17].

First, we formulate a temperature square error integrals minimization problem and theoretically derive a sensitivity function for this problem. The shape gradient functions are derived
theoretically using the Lagrange multiplier method and the formulation of the material derivative. In a similar manner, we can derive the shape gradient function for a prescribed problem where the time histories of temperature gradient distributions are specified in the prescribed subdomains. Then, the numerical method based on the traction method for these prescribed problems is presented. Lastly, the successful results for the two-dimensional problems demonstrate the validity of the presented method.

2. Temperature Square Error Integrals Minimization Problem

2.1 Formulation of problem

We consider a domain $\Omega \in \mathbb{R}^n, (n = 2, 3)$, with boundaries $\Gamma = \Gamma_0 \cup \Gamma_1$. The temperature is represented by $\phi = \phi(x, t)$ in space $x \in \Omega$ and time $t \in [0, T]$. $R$ is the set of real numbers. The temperature $\phi(\Gamma_0, [0, T])$ distributed on $\Gamma_0$, the heat flux $q(\Gamma_1, [0, T])$ distributed on $\Gamma_1$, the heat conductivity coefficient $k$, density $\rho$, capacity $c$, and heat source $f(x, t)$ distributed in $\Omega$, and the initial temperature distribution $\phi(x, 0) = \phi_0(x)$ are given as known functions or values.

A domain variation problem where the temperature distribution $\phi(\Gamma_D, [0, T])$ is specified to be $\phi_D(\Gamma_D, [0, T])$ on subboundaries $\Gamma_D \subseteq \Gamma_1$ can be regarded as a shape determination problem, as shown in Fig. 1. The domain variation of the heat-conduction field from domain $\Omega$ to $\Omega_s = \overrightarrow{T}_s(\Omega)$ can be described using a one-to-one mapping $\overrightarrow{T}_s$. The index $s$ denotes the history of domain variation. $\overrightarrow{T}_s(\Omega)$ is assumed to be the elements of a set $D$ of admissible functions which satisfy the restrictions on domain variation. For simplicity, we assume that the subboundaries $\Gamma_1$ are invariable, that is, $\overrightarrow{T}_s(\Gamma_1) = \Gamma_1$. This problem is formulated as

\begin{align*}
\text{given } & \Omega \text{ and } \\
k, \rho, c, f, q, \phi, \phi_0: & \text{ fixed in space } \quad (1) \\
\text{find } & \Omega_s \text{ or } \overrightarrow{T}_s(\Omega) \in D \quad (2)
\end{align*}

Fig. 1. Determination problem of domain with prescribed temperature on prescribed subboundary $\Gamma_D \subset \Gamma_1$. 

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that minimize \( \int_0^T E_{\Gamma_D} dt \) \hspace{1cm} (3)

subject to \( \int_0^T \left\{ a(\phi, w) + b(\phi, w) - c(w) - d(w) \right\} dt = 0 \quad \phi \in \Psi_t, \forall w \in W_t \) \hspace{1cm} (4)

where the square error integrals \( E_{\Gamma_D} \) are defined by

\[
E_{\Gamma_D} = E_{\Gamma_D}(\phi - \phi_D, \phi - \phi_D) dt = \int_{\Gamma_D} (\phi - \phi_D)^2 d\Gamma \tag{5}
\]

Equation (4) is the variational form, or the weak form using the adjoint temperature \( w(\vec{x}, t) \), of the state equation. The bilinear forms \( a(\phi, w) \) and \( b(\phi, w) \) and the linear forms \( c(w) \) and \( d(w) \) are defined by

\[
a(\phi, w) = \int_{\Omega} k\phi w_j dx, \quad b(\phi, w) = \int_{\Omega} \rho c\phi w dx, \\
c(w) = \int_{\Omega} fwdx, \quad d(w) = \int_{r_1} qwd\Gamma
\]

The sets \( \Psi_t \) and \( W_t \) of the admissible temperature \( \phi \) and adjoint temperature \( w \) are given by

\[
\Psi_t = \{ \phi(\vec{x}, t) \in H^1(\Omega_s \times [0, T]) | \phi(\vec{x}, t) = \phi^0(\vec{x}, t), \forall t \in [0, T], \vec{x} \in \vec{T}_s(\Gamma_0) \}, \\
k\phi(\vec{x}, t) w_j = q(\vec{x}, t), \forall t \in [0, T], \vec{x} \in \vec{T}_s(\Gamma_1), \phi(\vec{x}, 0) = \phi_0(\vec{x}), \vec{x} \in \Omega_s \} \tag{6}
\]

\[
W_t = \{ w(\vec{x}, t) \in H^1(\Omega_s \times [0, T]) | w(\vec{x}, t) = 0, \forall t \in [0, T], \vec{x} \in \vec{T}_s(\Gamma_0), w(\vec{x}, T) = w_T(\vec{x}) = 0, \vec{x} \in \Omega_s \} \tag{7}
\]

In the tensor notation of this paper, the Einstein summation convention and partial differential notations \( \partial(\cdot)/\partial x_i \) and \( \partial(\cdot)/\partial t \), respectively, are used. \( \mathcal{H}^n(\cdot) \) and \( \vec{n} \) denote the Sobolev space and the outward unit normal vector, respectively.

### 2.2 Shape gradient function

Applying the Lagrange multiplier method, this problem can be rendered into a stationary problem of the Lagrange functional

\[
L = \int_0^T \left\{ E_{\Gamma_D}(\phi - \phi_D, \phi - \phi_D) \right\} dt - \int_0^T \left\{ a(\phi, w) + b(\phi, w) - c(w) - d(w) \right\} dt \tag{8}
\]
The derivative $\dot{L}$ with respect to $s$ is derived using the velocity function $\mathbf{V}(\Omega_s) = \partial T_s(\Omega)/\partial s = \partial \overrightarrow{T_s}(\Omega_s)/\partial s$ [16, 17] as

\[
\dot{L} = -\int_0^T \{a(\phi, w') + b(\phi_s, w') - c(w') - d(w')\} dt
\]

\[
-\int_0^T \{a(\phi', w) + b(\phi_s', w) - 2E_{\Gamma_p}(\phi', \phi - \phi_D)\} dt + l_C(\mathbf{V})
\]  

(9)

where $\cdot$ is the material derivative, and $\cdot'$ is the shape derivative for domain variation of the distributed function under a spatially fixed condition. The linear form $l_C(\mathbf{V})$ of the velocity function $\mathbf{V}$ is given by

\[
l_C(\mathbf{V}) = \int_{\Gamma_0} Gn \cdot \mathbf{V} d\Gamma
\]

(10)

\[
G = \int_0^T \{-k\phi_j w_j\} dt
\]

(11)

From Eq. (9), $\phi(\vec{x}, t)$ and $w(\vec{x}, t)$ are obtained by

\[
\int_0^T \{a(\phi, w') + b(\phi_s, w') - c(w') - d(w')\} dt = 0 \quad \forall w' \in W_t
\]

(12)

\[
\int_0^T \{a(\phi', w) + b(\phi_s', w) - 2E_{\Gamma_p}(\phi', \phi - \phi_D)\} dt = 0 \quad \forall \phi' \in \Psi_t
\]

(13)

which indicate the variational form of the original state equation for temperature $\phi(\vec{x}, t)$ and the variational form for $w(\vec{x}, t)$, which we call an adjoint equation. Under the conditions satisfying Eqs. (12) and (13), the derivative of the Lagrange functional agrees with that of the objective functional and the linear form $l_C(\mathbf{V})$ with respect to $\mathbf{V}$:

\[
\dot{L}_{\phi,w} = \left(\dot{\mathbf{E}}_{\Gamma_p} dt\right)_{\phi,w} = l_C(\mathbf{V})
\]

(14)

The coefficient vector function $Gn$ in Eq. (10) represents a sensitivity function relative to domain variation and is the so-called shape gradient function. The scalar function $G$ is called the shape gradient density function. The shape gradient function can be derived theoretically, thus, domain variation can be analyzed by the traction method [16, 17].
3. Temperature Gradient Square Error Integrals Minimization Problem

In the previous section, we formulated a prescribed problem for time histories of temperature distributions \( \phi(\Gamma^D, [0, T]) \) on the prescribed subboundaries \( \Gamma^D \subseteq \Gamma_1 \) in unsteady heat-conduction fields and derived the shape gradient function for the problem. In a similar manner, we can derive the shape gradient function for a prescribed problem where the time histories of temperature gradient distributions \( \nabla \phi(\Omega^D, [0, T]) \) are specified by \( g^D(\Omega^D, [0, T]) \) in the prescribed subdomains \( \Omega^D \subset \Omega \), as shown in Fig. 2.

3.1 Formulation of problem

This problem is formulated as

\[
\text{given } \Omega \text{ and } k, \rho, c, \beta, \phi^0; \text{ fixed in space} \quad (15)
\]

\[
\text{find } \Omega_s \text{ or } T_s(\Omega) \in D \quad (16)
\]

\[
\text{that minimize } \int_0^T E_{\Omega_s} dt \quad (17)
\]

\[
\text{subject to } \int_0^T \left\{ a(\phi, w) + b(\phi, w) - c(w) - d(w) \right\} dt = 0 \quad \phi \in \Psi, \quad \forall w \in W_i \quad (18)
\]

where the square error integrals \( E_{\Omega_s} \) are defined by

\[
E_{\Omega_s} = E_{\Omega_s}(\nabla \phi - \nabla \phi^D, \nabla \phi - \nabla \phi^D) = \int_{\Omega_s} (\nabla \phi - \nabla \phi^D) \cdot (\nabla \phi - \nabla \phi^D) dx \quad (19)
\]

![Fig. 2. Determination problem of domain with prescribed temperature gradient in prescribed subdomain \( \Omega^D \subset \Omega \).](image-url)
For simplicity, we assumed that the subboundaries $\Gamma_0$ and the prescribed subdomains $\Omega_D$ are invariable, that is, $\tilde{T}_s(\Gamma_0) = \Gamma_0$ and $\tilde{T}_s(\Omega_D) = \Omega_D$.

3.2 Shape gradient function

In a manner similar to the previous section, the derivative $\hat{L}$ is derived as \cite{16, 17}

$$
\hat{L} = \int_0^T \{ a(\phi, w') + b(\phi_j, w') - c(w') - d(w') \} dt \\
- \int_0^T \{ a(\phi', w) + b(\phi'_j, w) - 2E_{\Omega}_D(\vec{\nabla}\phi', \vec{\nabla}\phi - \vec{g}_D) \} dt + l_G(\vec{V})
$$

(20)

The linear form $l_G(\vec{V})$ of the velocity function $\vec{V}$ and the shape gradient function for this problem are obtained in the same manner as in the previous section by

$$
l_G(\vec{V}) = \int_{\Gamma_1} G n \cdot \vec{V} d\Gamma
$$

(21)

$$
G = \int_0^T \{ -k\phi_j w_j - \rho c\phi_j w + f w + \nabla_n(qw) + (qw)\kappa \} dt
$$

(22)

where $\nabla_n(\cdot) \equiv \nabla(\cdot) \cdot n$ and $\kappa$ denote the mean curvature. The distribution of temperature $\phi(\vec{x}, t)$ and the adjoint temperature $w(\vec{x}, t)$ are obtained by

$$
\int_0^T \{ a(\phi, w') + b(\phi_j, w') - c(w') - d(w') \} dt = 0 \quad \forall w' \in W_t
$$

(23)

$$
\int_0^T \{ a(\phi', w) + b(\phi'_j, w) - 2E_{\Omega}_D(\vec{\nabla}\phi', \vec{\nabla}\phi - \vec{g}_D) \} dt = 0 \quad \forall \phi' \in \Psi_t
$$

(24)

Under the condition satisfying Eqs. (23) and (24), the derivative of the Lagrange functional agrees with that of the objective functional and the linear form $l_G(\vec{V})$ with respect to $\vec{V}$:

$$
\dot{L}_{\phi, w} = \left( \int_{\Omega_D} dt \right)_{\phi, w} = l_G(\vec{V})
$$

(25)
4. Numerical Solution Technique

4.1 Traction method

The traction method is a procedure for determining the amount of domain variation (velocity function $\mathbf{V}$) that reduces the objective functional, based on the governing equation (26). This method uses the gradient method in a Hilbert space, a technique that is also employed in distributed parameter optimal control problems.

$$a^E(\mathbf{V}, \mathbf{y}) = -l_G(\mathbf{y}), \quad \forall \mathbf{y} \in D \tag{26}$$

where the bilinear form $a^E(\mathbf{V}, \mathbf{y})$ gives the strain energy of the pseudolinear elastic body defined in the domain $\Omega$, as

$$a^E(\mathbf{u}, \mathbf{v}) = \int_{\Omega} A_{ijkl} u_{i,j} v_{l,k} \, dx \tag{27}$$

where $\mathbf{u}$, $\mathbf{v}$, and $A_{ijkl}$ are displacement, variational displacement, and an elastic tensor, respectively.

The governing equation (26) indicates that the velocity field $\mathbf{V}$ is a displacement field when negative shape gradient functions $-G \mathbf{n}$ act on the boundaries as external forces. In other words, using the traction method, domain variation is a displacement field when the shape gradient functions act as external forces in a pseudoelastic problem. Accordingly, Eq. (26) can be solved using a solution to ordinary linear-elastic problems, thus confirming the general applicability of the traction method. In this study, the finite-element method is used. The elastic tensor is given for simplicity as

$$A_{ijkl} = \delta_{ik} \delta_{jl} \tag{28}$$

4.2 Numerical procedure

A flowchart of the shape optimization system is shown in Fig. 3. The main elements of the system include two unsteady heat conduction analyses, calculation of the shape gradient function, a velocity analysis based on the traction method, and shape updating. The shape gradient function is evaluated using the two heat conduction analyses in which the distributions of temperature $\phi(\mathbf{x}, t)$ and adjoint temperature $w(\mathbf{x}, t)$ are analyzed. The time histories of temperature distributions are evaluated using the finite-element method for a space integral and the Crank–Nicolson method for a time integral. In calculating the solution to the original state equation (12) or (23), the temperature $\phi(\mathbf{x}, t)$ is analyzed using the initial condition $\phi(\mathbf{x}, 0) = \phi_0(\mathbf{x})$ for time from $t = 0$ to $t = T$. On the other hand, in calculating the solution to the adjoint equation (13) or (24), the adjoints temperature $w(\mathbf{x}, t)$ is analyzed using the initial condition $w(\mathbf{x}, T) = w_T(\mathbf{x}) = 0$ for time from $t = T$ to $t = 0$. Then, the shape gradient function is calculated using the results. The domain variation $\mathbf{V}$ of Eq. (26) is analyzed using the finite-element method. Shape optimization analysis is performed by repeatedly executing these elements sequentially. To perform an analysis, a domain variation coefficient $\Delta s$ is set which adjusts the magnitude of the domain variation $\Delta s \mathbf{V}$ per iteration.
5. Numerical Results

We present the results of two numerical analyses for 2D shape determination problems using the traction method and shape gradient function derived in previous sections.

5.1 Coolant flow passage in a wing

We analyzed a shape determination problem of coolant flow passage in a wing for a temperature prescribed problem, as shown in Fig. 4. The outer surface boundary of the wing was assumed as the prescribed subboundary $\Gamma_D = \Gamma_1$. The shape shown in Fig. 5 was chosen as the objective shape, and the temperature distribution over the outer surface boundary was assumed to be the prescribed temperature distribution $\phi_D(x, t)$. The design boundary is the left-side boundary of coolant flow passage. For simplicity, the case we considered had the following conditions: length of wing $l = 0.25$ m, specified temperature $\phi = 20$ deg, heat-conductivity coefficient $k = 0.204$ kW/m deg, heat flux $q = 150$ kW/m$^2$, density $\rho = 2710$ kg/m$^3$, capacity $c = 0.896$ kJ/kg deg, initial temperature $}\phi_0\).
distribution $\phi_0(x) = \phi_0 = 20$ deg, and specified period of time $T = 0.05$ s. By dividing the time interval into 100 steps, the Crank–Nicolson method was applied. The heat source $f$ was disregarded.

Numerical results are shown in Figs. 6 to 9. Figure 6 shows a comparison of the shapes and temperature distributions between the initial domain and the converged domain. Figure 7 shows the finite-element meshes. Figure 8 shows the time histories of temperature at point A in Fig. 4. Figure 9 shows the iterative history ratio of the objective functional normalized with the initial value. From these results, we observe a slight difference between the objective domain and converged domain. It is possible to regard the set shape identification problem as a problem with strong multicrestedness. Therefore, it is dependent on the chosen shape of the initial domain, and the analyzed converged domain seems to be one localized solution. However, it was confirmed that the time histories of temperature in the converged domain agreed with those in the objective domain, and the value of the objective functional approached zero. According to the numerical results of this basic problem, we
confirmed the validity of the present method. This numerical calculation required about 1 hour of CPU time on a single Alpha21164A/500 MHz processor.

5.2 Nozzle

In the prescribed problem for temperature gradient distribution, we analyzed a nozzle thickness shape problem as shown in Fig. 10. Considering that the two halves are symmetrical, the upper half of domain A–B–C–D was analyzed. The subdomain in the neighborhood of the inner wall of the nozzle was assumed as the prescribed subdomain $\Omega_D$. The shape and temperature gradient distributions shown in Fig. 11 were chosen as the objective shape and the temperature gradient

Fig. 7. Finite-element meshes in the coolant flow passage problem.

Fig. 8. Time histories of temperature in the coolant flow passage problem at the left-hand-side point in Fig. 4.

Fig. 9. Iterative history of objective functional in the coolant flow passage problem.
Fig. 10. Shape determination problem of a nozzle.

Fig. 11. Objective temperature gradient distributions in the nozzle problem.

Fig. 12. Results of temperature gradient distributions in the nozzle problem.
distributions, respectively. The design boundary is the outside boundary A–D. The case we considered had the following conditions: length of wing \( l = 0.06 \) m, specified temperature \( \phi = 100 \) deg, heat-conductivity coefficient \( k = 0.204 \) kW/m deg, heat flux \( q = 0 \) kW/m^2, density \( \rho = 2710 \) kg/m^3, capacity \( c = 0.896 \) kJ/kg deg, initial temperature distribution \( \phi_0(x) = \phi_0 = 20 \) deg, and specified period of time \( T = 0.001 \) s. In the time integral, by dividing the time period into 100 steps, the Crank–Nicolson method was applied. The heat source \( f \) was disregarded.

Numerical results of this problem are shown in Figs. 12 to 15 in a similar manner to the above-mentioned results. We confirmed that the value of the objective functional approached zero,

Fig. 13. Finite-element meshes in the nozzle problem.

Fig. 14. Time histories in the nozzle problem of temperature gradient in the \( x_5 \) direction at point P in Fig. 10.

Fig. 15. Iterative history of objective functional in the nozzle problem.

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and the converged domains analyzed by the proposed method exhibited good agreement with the objective domain. Based on these results, we confirmed the validity of the present method. This numerical calculation required about 20 minutes of CPU time on a single Alpha21164A/500MHz processor.

6. Conclusions

In this paper, we derived the shape gradient functions with respect to the shape identification problems of unsteady heat-conduction fields to control temperature distributions and temperature gradient distributions to prescribed distributions. The validity of the traction method using the derived shape gradient functions was confirmed by the numerical results.

Literature Cited


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